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REFLECTION OF PLANE WAVES BY THE FREE BOUNDARY OF A POROUS ELASTIC HALF-SPACE

M. CIARLETTA

D.I.I.M.A., Università di Salerno, Via Ponte Don Melillo, 84084 *Fisciano, Salerno, Italy. E-mail: ciarlett@diima.unisa.it*

AND

M. A. SUMBATYAN

Research Institute of Mechanics and Applied Mathematics, Stachki Prospect 200/1, *Rostov-on-Don* 344090, *Russia. E-mail: sumbat@math.rsu.ru*

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In the present paper, we study reflection of inclined incident plane waves from a free boundary of the half-space in which the material is described by constitutive equations valid for elastic solids with voids. Both the cases of the transverse and longitudinal incident waves are considered, and it is shown that only the transverse one can propagate in the solid without attenuation, after having been reflected from the free boundary surface. The reflection coefficient and the amplitude of the surface oscillations are expressed in explicit form. The general results are demonstrated for several hypothetical porous materials, and it is shown that the reflection coefficient and the vibration amplitude are typically less than in classical media without voids. However, for relatively large transverse wave speed and high porosity, free boundary oscillation can exceed the classical one.

1. INTRODUCTION

Investigation of dynamic properties of various elastic solids is a very important problem in the practice of ultrasonic inspection of materials, vibrations of engineering structures, in seismology and many other fields. Usually, such materials can be adequately described by classical dynamic equations of linear isotropic elastic solids [1]. However, some materials of a more complex microstructure (composite materials, granular materials, soils, etc.) show specific characteristic response to an applied dynamic load.

There are a number of theories which describe mechanical properties of porous materials, and the most known of them is a Biot consolidation theory of fluid-saturated porous solids [2, 3]. Typically, these theories reduce to classical elasticity when the pore fluid is absent. This is why Cowin and Nunziato proposed a new theory to more adequately describe the dynamic nature of homogeneous elastic materials with voids free of fluid [4]. A general theory of such materials has been currently well developed by many authors (see, for example, references [5, 6, 7]), but only a few concrete problems have been solved, which does not permit real estimates of practical merits of this model.

Generally, this theory is founded on the balance of energy, when assuming that presence of the pores involves an additional degree of freedom, namely, the fraction of elementary volume. As a consequence, the bulk mass density is given by the product of two fields, the

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void volume fraction and the mass density of the matrix (elastic) material. Maybe, the main disadvantage of the Cowin–Nunziato theory is that the physical parameters that enter the governing equations of motion have never been measured, and the authors usually operate with hypothetical sets of the parameters. However, it should be noted that precise measurement of physical constants is a great problem also in other theories (in fact, even in classical linear elasticity elastic moduli can, as a rule, be measured with an error of several per cent, which makes absurd results often obtained as a calculation with four to five significant digits).

In the framework of the Cowin–Nunziato theory, Puri and Cowin [8] first investigated all possible types of plane waves which can propagate in linear elastic materials with voids. They discovered dispersion and dissipation (attenuation) of these waves with distance. More recently Chandrasekharaiah [9] studied the influence of the voids' volume fraction on Rayleigh surface waves.

The main aim of the present work is to study reflection and mode conversion of plane waves (both transverse and longitudinal), incident at the angle θ on the free boundary surface of the porous elastic half-space. We compare the results obtained with the case of classical elastic materials, and give an estimate of the difference for the reflection angles and the amplitude of the free surface vibrations. Similar problems have been investigated, in frames of the consolidation theory, in references [10–12].

2. SURVEY OF PREVIOUS RESULTS

The theory of linear isotropic elastic materials with voids, in the case of harmonic oscillations with the angular frequency Ω, involves the following equations of motion [4, 8]:

$$\mu\Delta\bar{u} + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\bar{u} + \beta\,\mathrm{grad}\,\phi + \rho\,\Omega^2\bar{u} = 0, \tag{1a}$$

$$\alpha\Delta\phi - \xi\phi - \beta\,\mathrm{div}\,\bar{u} + (\mathrm{i}\omega\Omega + \rho k\Omega^2)\phi = 0. \tag{1b}$$

Here $\bar{u} = \{u_1, u_2, u_3\}$ is the displacement vector; $\phi = v - v_0$ the change in the volume fraction from the reference volume fraction; λ and μ the classical elastic moduli; ρ the density of the material; and α, β, ξ, ω, k the material coefficients related to porosity. All physical quantities contain the time-dependent factor $\exp(-\mathrm{i}\Omega t)$, which is omitted later on. It is obvious that in the case $\beta = 0$ the elastic displacement field $\bar{u}$ and the "porosity" ϕ can be separately determined from equations (1a) and (1b) respectively.

As soon as the functions $\bar{u}$ and ϕ are defined, the components of the stress tensor are given by the constitutive equations as follows (δ_{ij} is Kronecker's delta),

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} + \beta\phi\delta_{ij}, \quad \varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \tag{1c}$$

with summation on the repeating index.

The boundary conditions at the free surface of the half-space $y \geq 0$ are (see Figure 1)

$$\sigma_{yy} = \sigma_{xy} = \sigma_{yz} = \partial\phi/\partial y = 0, \quad y = 0. \tag{2}$$

Let (x, y) be the vertical plane and the axis z be orthogonal (i.e., horizontally directed) to the plane. Then the simplest solution of the system (1)

$$\bar{u} = \{0, 0, w(x, y)\}, \quad \phi = 0, \tag{3}$$

reduces equations (1) to the Helmholtz equation

$$\Delta w + k_s^2 w = 0, \quad k_s = \Omega/c_s > 0, \quad c_s^2 = \mu/\rho. \tag{4}$$

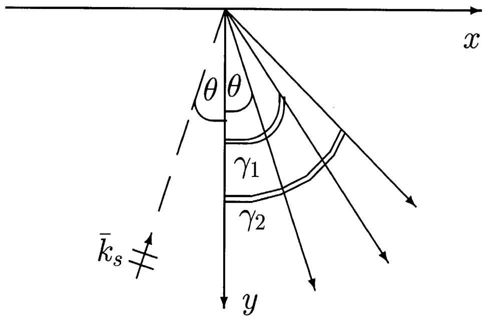

Figure 1. Incidence of the plane vertically polarized transverse wave onto the free boundary of the porous elastic half-space.

So-called plane waves of horizontal polarization

$$w(x, y) = e^{ik_s(x \sin \gamma + y \cos \gamma)} \tag{5}$$

are natural solutions of equation (4) with arbitrary (real-valued) angle of propagation γ. Thus, plane waves of horizontal polarization (1) do not decay with distance, (2) are non-dispersive, and (3) propagate with the classical transverse ("shear") wave speed c_s.

In the case of vertical polarization,

$$\bar{u} = \{u_x(x, y), u_y(x, y), 0\}, \quad \phi = \phi(x, y), \tag{6}$$

Chandrasekharaiah [9] has shown that equations (1) are automatically satisfied if the classical wave potentials p and q,

$$u_x = \frac{\partial p}{\partial x} - \frac{\partial q}{\partial y}, \quad u_y = \frac{\partial p}{\partial y} + \frac{\partial q}{\partial x}, \tag{7}$$

satisfy the following equations:

$$\Delta q + k_s^2\, q = 0, \tag{8}$$

$$\left[(\Delta + k_p^2)\left(\Delta - \frac{1}{l_2^2} + \frac{i\omega}{\alpha}\Omega + \frac{\rho k}{\alpha}\Omega^2\right) + \frac{H}{l_1^2}\Delta\right]p = 0, \tag{9}$$

where

$$k_p = \frac{\Omega}{c_p}, \quad c_p^2 = \frac{\lambda + 2\mu}{\rho}, \quad H = \frac{\beta}{\lambda + 2\mu}, \quad l_1^2 = \frac{\alpha}{\beta}, \quad l_2^2 = \frac{\alpha}{\xi}, \tag{10}$$

and c_p is the well-known longitudinal ("pressure") wave speed. For all this, the function ϕ is determined from the equation

$$-H\phi = \Delta p + k_p^2 p. \tag{11}$$

Equation (8) coincides with equation (4). Therefore, a non-dispersive transverse wave of vertical polarization can propagate without attenuation along arbitrary directions in the plane (x, y). It is a very interesting and important feature of the wave process in porous elastic media that the plane transverse wave (both of horizontal and vertical polarization), being non-dispersive, propagates in the medium without any decay.

Solution of equation (9) for longitudinal potential $p(x, y)$ is of a more complex form, and this will be discussed in detail in the next section.

3. REFLECTION OF THE TRANSVERSE PLANE WAVE FROM THE FREE BOUNDARY

So far as the case of the transverse wave of horizontal polarization involves only single equation (4), its reflection from a free boundary, under an arbitrary angle of incidence θ, is described absolutely in the same way as in classical linear elasticity [13]. In particular, the reflection angle γ is equal to the angle of incidence: $\gamma = \theta$ and there is no mode conversion of the incident transverse wave to the longitudinal one.

Now we pass to the shear wave of vertical polarization. Let the incident plane wave

$$q_{inc} = e^{ik_s(x \sin\theta - y\cos\theta)}, \quad p_{inc} = 0, \tag{12}$$

fall onto the free boundary $y = 0$ of the porous elastic half-space, with the angle of incidence θ (see Figure 1).

Since the shear component of the wave field is subjected to the same reflection law as in a classical medium, the total transverse potential q can be represented as

$$q = e^{ik_s(x \sin\theta - y\cos\theta)} + R_q e^{ik_s(x\sin\theta + y\cos\theta)}, \tag{13}$$

where R_q is the reflection coefficient. One may seek the longitudinal potential p in the form

$$p = T\, e^{i\chi(x\sin\gamma + y\cos\gamma)}, \tag{14}$$

where both the coefficient T and the angle γ should be defined from the boundary conditions. By substituting representation (14) into equation (9), one comes to the equation which predetermines the unknown value of the wave number χ,

$$(k_p^2 - \chi^2)(-l_2^2\chi^2 - 1 + i\omega^* k_p + k^{*2}k_p^2) - N\chi^2 = 0, \tag{15}$$

where $0 < N = (l_2^2/l_1^2)H < 1$ is a dimensionless parameter [4, 8], and the coefficients l_2, $\omega^* = \omega l_2^2 c_p/\alpha$ and $k^* = l_2\, c_p\sqrt{\rho k/\alpha}$ are all of length dimension.

Equation (15) is a biquadratic equation with respect to the wave number χ. Obviously, only in the case $N = 0 \Rightarrow \beta = 0$ (separate elastic and porosity wave fields) this admits the solution $\chi = k_p$ corresponding to propagation of classical longitudinal wave with the wave speed c_p. Generally, equation (15) is equivalent to

$$l_2^2\chi^4 + \chi^2[1 - N - i\omega^* k_p - (l_2^2 + k^{*2})k_p^2] - k_p^2(1 - i\omega^* k_p - k^{*2}k_p^2) = 0. \tag{16}$$

Equation (16) has been studied in detail by Puri and Cowin [8], for various ranges of physical parameters. Generally, solution of equation (16) is complex-valued; however, this admits real-valued solutions for the cases of limitly low and limitly high frequencies. In the first case, when $l_2 k_p \gg 1$ $(\Rightarrow \omega^* k_p,\ k^* k_p \gg 1)$, equation (16) is equivalent to (we must keep main asymptotic terms both in real and imaginary parts)

$$l_2^2\chi^4 - [(l_2^2 + k^{*2})k_p^2 + i\,\omega^* k_p]\chi^2 + (k^{*2}k_p^2 + i\,\omega^* k_p)k_p^2 = 0, \tag{17}$$

whose two solutions are real-valued (of course, we use only principal value $\mathrm{Re}(\sqrt{x}) \geq 0$, when applying root squares),

$$\chi_1 = k_p = \frac{\Omega}{c_p}, \quad \chi_2 = k_3 = \frac{\Omega}{c_3} + \frac{i\,\omega^*}{2l_2k^*} = \frac{\Omega}{c_3} + \frac{i\,\omega\, c_3}{2\alpha}, \quad c_3 = \frac{l_2\, c_p}{k^*} = \sqrt{\frac{\alpha}{\rho k}}, \tag{18}$$

in accordance with notations [8] related to the wave phase speed c_3.

Unfortunately, high-frequency oscillations are of less importance in real practice. In fact, $l_2\, k_p = 2\pi l_2/\lambda_p$, where λ_p is the length of the longitudinal wave. Since the parameter l_2, with dimension of length, is coupled with the microstructure of the medium, this seems to be rather small when compared with the wavelength λ_p, so $l_2\, k_p$ is apparently a small dimensionless parameter. Only in acoustic microscopy, when the length of the ultrasonic wave is of the order of particles' size, this parameter can be of the order of 1.

In the more realistic low-frequency case $l_2 k_p \ll 1$ (that is well known to correspond to real situation in seismic wave propagations) the asymptotic solution of equation (16) is [8]

$$\chi_1 = \frac{k_p}{\sqrt{1-N}} = \frac{\Omega}{c_p\sqrt{1-N}}, \tag{19a}$$

$$\chi_2 = \frac{\omega^* k_p}{2l_2\sqrt{1-N}} + \frac{\mathrm{i}\sqrt{1-N}}{l_2} = \frac{\Omega}{c_4} + \frac{\mathrm{i}\sqrt{1-N}}{l_2}, \quad c_4 = \frac{2\alpha\sqrt{1-N}}{\omega l_2}, \tag{19b}$$

where the wave phase speed c_4 is introduced in reference [8].

Some important conclusions may be extracted from equations (18) and (19): (1) high-frequency propagation in the porous medium admits a classical longitudinal wave speed c_p (for $\gamma = \gamma_1$); (2) the undamped low-frequency longitudinal wave with $\gamma = \gamma_1$ possesses a wave speed $c_p\sqrt{1-N}$ which is different from c_p; (3) the second reflected wave (with $\gamma = \gamma_2$) always decays with distance, both for high and low frequencies; but more wonderful is that at $\Omega \to 0$ and $\Omega \to \infty$ the respective attenuation coefficients $\omega c_3/2\alpha$ and $\sqrt{1-N}/l_2$ do not vanish, and moreover these do not depend on the frequency Ω at all. Since l_2 is a small parameter with a dimension of length the attenuation coefficient in the low-frequency case $\sqrt{1-N}/l_2$ is considerably large, so this longitudinal wave, after having been reflected from the boundary, decays almost suddenly. However, this wave "participates" when satisfying boundary conditions, with the same degree-like waves with the wave number k_s (both incident and reflected ones) and the first reflected longitudinal wave with $\chi = \chi_1$.

It should be noted that Puri and Cowin [8] have proved that the wave number χ_1 represents predominantly an elastic wave and χ_2 represents a predominantly volume fraction wave.

Generally, the total structure of the wave field in the half-space $y > 0$ can be expressed in the following way:

$$q = \mathrm{e}^{\mathrm{i}k_s(x\sin\theta - y\cos\theta)} + R_q\,\mathrm{e}^{\mathrm{i}k_s(x\sin\theta + y\cos\theta)}, \tag{20a}$$

$$p = T_1\,\mathrm{e}^{\mathrm{i}\chi_1(x\sin\gamma_1 + y\cos\gamma_1)} + T_2\,\mathrm{e}^{\mathrm{i}\chi_2(x\sin\gamma_2 + y\cos\gamma_2)}. \tag{20b}$$

As usually, Snell's law implies

$$\chi_1\sin\gamma_1 = \chi_2\sin\gamma_2 = k_s\sin\theta, \tag{21}$$

which determines (generally, complex-valued) reflection angles γ_1 and γ_2. The only three remaining unknown coefficients R_q, T_1, T_2 should be defined by satisfying the null boundary conditions (2) for the components $\sigma_{xy}, \sigma_{yy}, \partial\phi/\partial y$ along the boundary line $y = 0$.

As follows from equations (1c) and (11) these boundary conditions, in terms of wave potentials p and q, are

$$\sigma_{xy} = 0 \;\Rightarrow\; 2\frac{\partial^2 p}{\partial x\partial y} + \frac{\partial^2 q}{\partial x^2} - \frac{\partial^2 q}{\partial y^2} = 0, \quad y = 0, \tag{22a}$$

$$\sigma_{yy} = \rho\left[c_p^2 \Delta p - 2c_s^2\left(\frac{\partial^2 p}{\partial x^2} - \frac{\partial^2 q}{\partial x\partial y}\right)\right] - \frac{\beta}{H}(\Delta p + k_p^2 p) = 0$$

$$\Rightarrow 2\left(\frac{\partial^2 p}{\partial x^2} - \frac{\partial^2 q}{\partial x\partial y}\right) + k_s^2 p = 0, \quad y = 0, \tag{22b}$$

$$\frac{\partial\phi}{\partial y} = 0 \Rightarrow \frac{\partial}{\partial y}(\Delta p + k_p^2 p) = 0, \quad y = 0. \tag{22c}$$

By using representations (20) for potentials one can reduce equations (22) to the following 3×3 system of linear algebraic equations with respect to the coefficients T_1, T_2, R_q:

$$T_1\chi_1^2 \sin(2\gamma_1) + T_2\chi_2^2 \sin(2\gamma_2) - R_q k_s^2 \cos(2\theta) = k_s^2 \cos(2\theta), \tag{23a}$$

$$T_1 \cos(2\theta) + T_2 \cos(2\theta) + R_q \sin(2\theta) = \sin(2\theta), \tag{23b}$$

$$T_1\chi_1 \cos\gamma_1 \,(k_p^2 - \chi_1^2) + T_2\chi_2 \cos\gamma_2 \,(k_p^2 - \chi_2^2) = 0. \tag{23c}$$

It is very interesting that the constants N, l_2, ω^*, k^* characterizing the presence of voids are not explicitly present in the last system (23). In fact, these coefficients are included only in the wave numbers χ_1, χ_2 and respective angles γ_1, γ_2.

4. PHYSICAL CONCLUSIONS FOR THE CASE OF TRANSVERSE INCIDENT WAVE

To begin with, let us prove that for any (finite) fixed frequency both the reflected longitudinal waves (with the wave numbers χ_1 and χ_2 respectively) are damped, and so decay when $y \to +\infty$. We give the proof by contradiction. If any χ^2, as a solution of equation (16), is positive, then by separating real and imaginary parts in expression (16), we can see that $\chi^2 = k_p^2$ (by setting the imaginary part to be equal to zero). But in the form (15), equivalent to (16), it becomes clear that $\chi^2 = k_p^2$ cannot be the exact solution of (16). Therefore, strictly speaking the *exact* value of χ^2 (and χ as well) is complex valued for any finite Ω (i.e., $\mathrm{Im}(\chi) \neq 0$). Then, due to Snell's law $\chi \sin\gamma = k_s \sin\theta$ and the quantity $\chi\cos\gamma = \sqrt{\chi^2 - k_s^2 \sin^2\theta}$ is always complex valued, so both the exponential terms in equation (20b) decay with the ordinate y increasing.

Thus, we come to the conclusion that only asymptotically for high and low frequencies, when the imaginary part of χ tends to zero, equation (16) admits real solutions $\chi = \chi_1$. The second solution $\chi = \chi_2$ always remains complex valued.

In both the asymptotic cases, the main physical parameter, the reflection coefficient R_q, can be expressed in explicit form. For high frequencies $\chi = k_p$ and equation (23) becomes equivalent to the system

$$T_1\chi_1^2 \sin(2\gamma_1) + T_2\chi_2^2 \sin(2\gamma_2) - R_q k_s^2 \cos(2\theta) = k_s^2 \cos(2\theta), \tag{24a}$$

$$T_1 \cos(2\theta) + T_2 \cos(2\theta) + R_q \sin(2\theta) = \sin(2\theta), \tag{24b}$$

$$T_2\chi_2 \cos\gamma_2 \,(k_p^2 - \chi_2^2) = 0, \tag{24c}$$

which determines R_q to be independent of the pair χ_2, γ_2 (recall that $\chi_1 \sin\gamma_1 = k_s \sin\theta$):

$$R_q = \frac{\chi_1 \cos\gamma_1 \,\mathrm{tg}^2(2\theta) - k_s \cos\theta}{\chi_1 \cos\gamma_1 \,\mathrm{tg}^2(2\theta) + k_s \cos\theta}, \quad \mathrm{tg} = \text{tangent}, \tag{25}$$

with

$$\chi_1 \cos\gamma_1 = \sqrt{\chi_1^2 - (\chi_1^2 \sin\gamma_1)^2} = \sqrt{k_p^2 - k_s^2 \sin^2\theta}, \tag{26}$$

a solution which coincides with the classical case of a medium without voids [13], that in view of $\chi_1 = k_p$ does not seem to be unexpected.

What is unexpected is just that the term $T_1\chi_1 \cos\gamma_1 (k_p^2 - \chi_1^2)$ in the system (23) always remains very small, (this implies T_2 to be very small too). We have performed numerous calculations, for various values of physical parameters (a FORTRAN code was used for this aim, which permits direct operation with complex-valued numbers), and the transformation coefficient T_2 never exceeded 0·02. Apparently, this is genetically based on the fact that the difference $k_p^2 - \chi_1^2$ is always small. Really, for high frequencies $\chi_1 \approx k_p$; for low frequencies $\chi_1 \approx k_p/\sqrt{1-N}$ which is very close to k_p (it is proved in reference [14] that $0 < N < 4c_s^2/3c_p^2$). Numerical analysis shows that $k_p^2 - \chi_1^2$ is relatively small for all Ω. If so, then for arbitrary Ω the system (24) is a good approximation for the system (23). This implies the representation (25) to be (approximately) uniformly valid, with corresponding χ_1 taken as an *exact* principal (predominantly elastic-wave-coupled) solution of equation (16), with respective value γ_1. To make a choice between two complex-valued quantities χ_1 and χ_2, which of them is indeed a predominantly elastic (i.e., principal) one, when solving equation (16), we accepted as γ_1 the value with $|\chi_1 - k_p| < |\chi_2 - k_p|$. Such an algorithm results in γ_1 always correctly. For all that relative error of approximation (25), when compared with the exact solution of the system (23), does not exceed 3%, and for the most part of physical parameters combinations it is less than 1%.

Obviously, approximation (25) yields an explicit formula for the low-frequency limit. Really, in this case $\chi_1 = k_p/\sqrt{1-N}$, so one can apply formula (25) for low frequencies by putting

$$\chi_1 \cos\gamma_1 = \sqrt{\frac{k_p^2}{1-N} - k_s^2 \sin^2\theta}. \tag{27}$$

It should be noted that both the high- and low-frequency reflection coefficients (25) with equations (26) and (27), respectively, do not depend on the frequency Ω. Among all physical parameters these depend only on the ratio c_p/c_s, the parameter N and, of course, upon the incident angle θ.

Figures 2–5 demonstrate behaviour of the reflection coefficient R_q versus frequency, for the four different hypothetical elastic materials with voids, considered by Puri and Cowin [8]. The following values are common for these materials: $c_p = 3873$ m/s, $c_s = 1937$ m/s, $\xi = 12$ GPa, $\beta = 10$ GPa, $H = 1/3$. All over these figures line 1 is related to the case $\theta = 10°$, line 2 to $\theta = 20°$, line 3 to $\theta = 25°$, line 4 to $\theta = 30°$, line 5 to $\theta = 35°$, line 6 to $\theta = 40°$.

We compare in these figures the results of direct numerical treatment (or, equivalently, results predicted by formula (25) with exact χ_1 and γ_1) with explicit asymptotic formulas. All horizontal straight lines represent classical solutions [13] for respective angle of incidence, which in accordance with the above-stated results coincide also with the high-frequency approximation. It can be seen from Figures 2–5 that all curves, after some

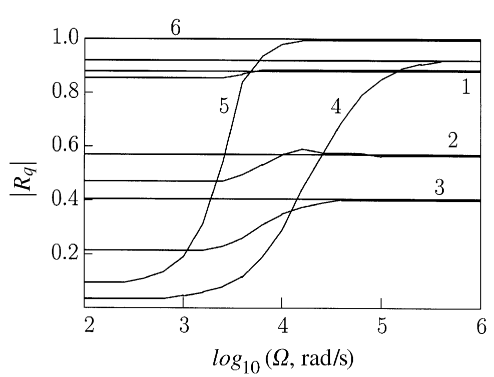

Figure 2. Reflection coefficient versus frequency for the first hypothetical porous material: $c_3 = 5000$ m/s, $c_4 = 16\,653$ m/s, $\omega = 1$ MPa s, $\alpha = 8$ GPa m^2.

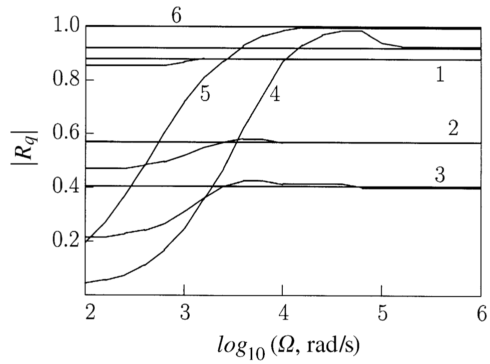

Figure 3. Reflection coefficient versus frequency for the second hypothetical porous material: $c_3 = 2000$ m/s, $c_4 = 1665$ m/s, $\omega = 10$ MPa s, $\alpha = 8$ GPa m^2.

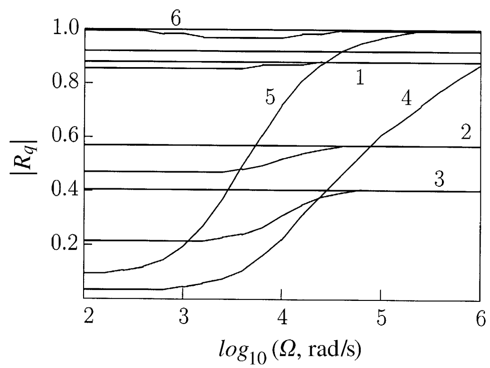

Figure 4. Reflection coefficient versus frequency for the third hypothetical porous material: $c_3 = 5000$ m/s, $c_4 = 1862$ m/s, $\omega = 1$ MPa s, $\alpha = 0{\cdot}1$ GPa m^2.

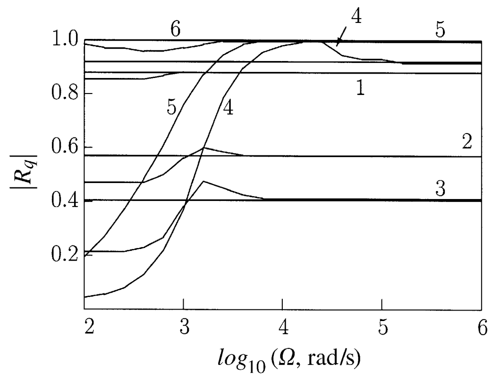

Figure 5. Reflection coefficient versus frequency for the fourth hypothetical porous material: $c_3 = 2000$ m/s, $c_4 = 3725$ m/s, $\omega = 10$ MPa s, $\alpha = 40$ GPa m^2.

oscillations, approach at high frequencies their own respective asymptotic horizontal line. For small frequencies, it is also clearly seen that at the left end of the frequency Ω variation, every curve again approaches its low-frequency asymptotic limit independent of the value of Ω.

From the graphics we can discover an interesting wave property of the porous media: despite the small difference between high-frequency $\chi_1 = k_p$ and low-frequency $\chi_1 = k_p/\sqrt{1-N}$ wave numbers, the respective values of R_q can differ considerably (see, for example, lines 4 and 5). The real reason of this feature is that corresponding values of $\chi_1 \cos\gamma_1$, being equal to $\sqrt{k_p^2 - k_s^2 \sin^2\theta}$ and $\sqrt{k_p^2/(1-N) - k_s^2 \sin^2\theta}$, respectively, for some values of θ are indeed far from each other. The range of θ, where such a behaviour takes place, depends of course on the ratio c_p/c_s and the value of N.

In the context of Figures 2–5 one can evidently notice that, as follows from system (25), for the case of normal incidence ($\theta = 0$) $R_q = -1 \Rightarrow |R_q| = 1$, like in a classical elastic medium [13]. It should also be noted that when the angle of incidence is greater than some critical value, the reflection coefficient $|R_q| = 1$ again, in agreement with classical theory [13].

5. FREE SURFACE OSCILLATIONS

Let us develop formulas for components of the vector of free boundary surface oscillations. It follows from equations (7) and (20) that

$$U_x = u_x|_{y=0} = \mathrm{i}\, k_s\, \mathrm{e}^{\mathrm{i}k_s\, x \sin\theta}[(T_1 + T_2)\sin\theta + (1 - R_q)\cos\theta]. \tag{28}$$

Since equation (23b) implies $T_1 + T_2 = (1 - R_q)\mathrm{tg}(2\theta)$, so equation (28), with the use of equation (25), gives

$$|U_x| = k_s \frac{\cos\theta}{\cos(2\theta)}|1 - R_q| = k_s \frac{2k_s \cos^2\theta/\cos(2\theta)}{|\chi_1 \cos\gamma_1\, \mathrm{tg}^2(2\theta) + k_s \cos\theta|}. \tag{29}$$

Further,

$$U_y = u_y|_{y=0} = \mathrm{i}\mathrm{e}^{\mathrm{i}k_s x \sin\theta}[T_1\chi_1\cos\gamma_1 + T_2\chi_2\cos\gamma_2 + (1 + R_q)k_s \sin\theta]. \tag{30}$$

One can obtain from equation (23a) that $T_1\chi_1\cos\gamma_1 + T_2\chi_2\cos\gamma_2 = (1 + R_q)k_s\cos(2\theta)/2\sin\theta$, hence equation (30) leads to

$$|U_y| = k_s\left[\frac{\cos(2\theta)}{2\sin\theta} + \sin\theta\right]|1 - R_q| = k_s \frac{|\chi_1\cos\gamma_1|\,\mathrm{tg}^2(2\theta)/\sin\theta}{|\chi_1\cos\gamma_1\,\mathrm{tg}^2(2\theta) + k_s\cos\theta|}. \tag{31}$$

General analysis of these expressions (29) and (31), on the basis of their analytical representations, is rather complicated. However, for the most importance in the seismic practice low-frequency case these permit estimates in explicit form. For small Ω we have $\chi_1 = k_p/\sqrt{1-N}$, so for θ less than the first critical angle $\chi_1\cos\gamma_1 = \sqrt{k_p^2/(1-N) - k_s^2\sin^2\theta} > k_p\cos\theta = \sqrt{k_p^2 - k_s^2\sin^2\theta}$, where the latter expression on the right-hand side corresponds to classical theory of elastic materials without voids. Therefore, it is clearly seen from equation (29) that $|U_x| < |U_x|_{classical}$. To make an estimate for $|U_y|$, we need to consider the behaviour of the function $f(\zeta) = \zeta/(\zeta + a)$, with $\zeta = \chi_1\cos\gamma_1 > 0$ and $a = k_s\cos\theta > 0$. Obviously, $f'(\zeta) = a/(\zeta + a)^2 > 0$, hence $|U_y|$ is a monotonically increasing function of $\chi_1\cos\gamma_1$ and consequently $|U_y| > |U_y|_{classical}$.

Thus, the horizontal component of the oscillation vector is less than in the classical case, but the vertical component is larger. What can be said regarding the modulus of this vector, i.e., what is larger: $|U_x|^2 + |U_y|^2$ or the analogous quantity in the classical case? To answer this question, let us rewrite $|U_x|$ and $|U_y|$ as follows:

$$|U_x| = \frac{k_s}{\sin\theta\,\mathrm{tg}(2\theta)}\frac{k_s\cos\theta}{\chi_1\cos\gamma_1 + k_s\cos\theta/\mathrm{tg}^2(2\theta)} = \frac{k_s}{\sin\theta\,\mathrm{tg}(2\theta)}\frac{A}{\zeta + a}, \tag{32a}$$

$$|U_y| = \frac{k_s}{\sin\theta\,\mathrm{tg}(2\theta)}\frac{\chi_1\cos\gamma_1\,\mathrm{tg}(2\theta)}{\chi_1\cos\gamma_1 + k_s\cos\theta/\mathrm{tg}^2(2\theta)} = \frac{k_s}{\sin\theta\,\mathrm{tg}(2\theta)}\frac{B\zeta}{\zeta + a}, \tag{32b}$$

where

$$A = k_s\cos\theta, \quad B = \mathrm{tg}(2\theta), \quad a = \frac{k_s\cos\theta}{\mathrm{tg}^2(2\theta)}, \quad \zeta = \chi_1\cos\gamma_1. \tag{32c}$$

Obviously, the behaviour of the function $|U_x|^2 + |U_y|^2$ is predetermined by the function

$$g(\zeta) = \frac{A^2 + B^2\zeta^2}{(\zeta + a)^2} \Rightarrow g'(\zeta) = 2\frac{B^2 a\zeta^2 - A^2}{(\zeta + a)^3}. \tag{33}$$

In our problem

$$B^2 a\zeta^2 - A^2 = k_s\cos\theta\,(\chi_1\cos\gamma_1 - k_s\cos\theta)$$

$$= k_s\left[\sqrt{\frac{k_p^2}{k_s^2(1-N)} - \sin^2\theta} - \sqrt{1 - \sin^2\theta}\right], \tag{34}$$

so the positive sign of the square brackets corresponds to a monotonically increasing function $g(\zeta)$ (33). Otherwise, the function $g(\zeta)$ is monotonically decreasing.

For the four hypothetical elastic materials with voids considered above ($N = 0{\cdot}2778$, $k_p/k_s = c_s/c_p = 1/2$) we have $(k_p^2/k_s^2)/(1-N) = 0{\cdot}35 < 1$, so $g'(\zeta) < 0$ and $g(\zeta)$ is a monotonically decreasing function. Therefore, for all four materials,

$$|U_x|^2 + |U_y|^2 < \left(|U_x|^2 + |U_y|^2\right)_{classical}. \tag{35}$$

Let us study this question in the general case. Since $(k_p^2/k_s^2) = (1-2\nu)/(2-2\nu)$ (ν is the Poisson coefficient which is $0 < \nu < 0{\cdot}5$), one can see that $(k_p^2/k_s^2) < 1/2$. It is also proved in reference [14] that $0 < N < (4/3)\,k_p^2/k_s^2 < 2/3$. The critical value of N, when the expression in square brackets (34) changes its sign, is given as follows:

$$\frac{k_p^2}{k_s^2(1-N)} = 1 \Rightarrow \frac{1}{2} < N = 1 - \frac{k_p^2}{k_s^2} < 1. \tag{36}$$

Therefore, for some materials with a relatively high ratio c_s/c_p and the "porosity index" N being between 1/2 and 2/3, the function $g(\zeta)$ in equation (33) can be monotonically increasing. In that case the amplitude of the free boundary surface full oscillation becomes higher than for the case of materials when voids are absent.

6. LONGITUDINAL INCIDENT WAVE

This case is of less interest for real practice, because longitudinal wave generally cannot propagate without attenuation, decaying with distance exponentially. This wave thus cannot reach the free boundary when approaching from a far zone. Only in two asymptotic cases of high and low frequencies can this wave fall onto the boundary without visible damping.

For high frequencies, when $\gamma_1 = k_p$, the reflection process is subjected just to the same reflection law as in the classical case [13]. Let us study in more detail the opposite low-frequency regime.

Here, by analogy to equations (20), the wave potentials p and q can be represented as

$$p = e^{i\chi_1(x \sin \gamma_1 - y \cos \gamma_1)} + R_p\, e^{i\chi_1(x \sin \gamma_1 + y \cos \gamma_1)} + R_2\, e^{i\chi_2(x \sin \gamma_2 + y \cos \gamma_2)}, \tag{37a}$$

$$q = Q\, e^{ik_s(x \sin \theta + y \cos \theta)}, \tag{37b}$$

where γ_1 is the angle of incidence and R_p is the reflection coefficient.

Boundary conditions in the form (22) involve the following 3×3 linear algebraic system to define the three unknown constants R_p, R_2, Q,

$$R_p\chi_1^2 \sin(2\gamma_1) + R_2\,\chi_2^2 \sin(2\gamma_2) - Qk_s^2 \cos(2\theta) = \chi_1^2 \sin(2\gamma_1), \tag{38a}$$

$$R_p \cos(2\theta) + R_2 \cos(2\theta) + Q \sin(2\theta) = -\cos(2\theta), \tag{38b}$$

$$R_p\chi_1 \cos\gamma_1(k_p^2 - \chi_1^2) + R_2\chi_2 \cos\gamma_2(k_p^2 - \chi_2^2) = 0, \tag{38c}$$

just with the same principal matrix as system (23). The solution of this system, when the element $a_{31} = \chi_1 \cos\gamma_1(k_p^2 - \chi_1^2)$ is small, can be expressed as follows:

$$R_p = \frac{\chi_1 \cos\gamma_1\, \mathrm{tg}^2(2\theta) - k_s \cos\theta}{\chi_1 \cos\gamma_1\, \mathrm{tg}^2(2\theta) + k_s \cos\theta}, \tag{39}$$

which coincides with the reflection coefficient R_q for the case of the transverse incident wave (see equation (25)), as in classical theory [13]: $R_p = R_q$.

7. CONCLUSIONS

1. Reflection of oblique incident plane waves from a free boundary of a porous elastic half-space has been considered, in the frames of the Cowin–Nunziato linear theory of materials with voids. A uniform analytic representation (25) for the reflection coefficient is developed, where the quantities χ_1, γ_1 are given as a solution of equation (16). The relative error of this explicit representation is less than 2%.
2. Generally, wave reflection from the free boundary is frequency dependent. However, at low and high frequencies, R_q does not depend on the frequency value; moreover, at the high-frequency limit the quantity R_q approaches the respective value given by a classical linear elastic theory.
3. As can be seen from Figures 2–5, "the rate of the approach" to respective limiting non-dispersive (i.e., constant with respect to Ω) values strictly depends on the set of physical parameters. Thus, for material represented in Figure 3, R_q remains frequency dependent also for very low frequencies, and in Figure 4 for very high frequencies, since over these ranges respective graphs for the function $|R_q(\Omega)|$ are not horizontal.
4. For both limiting cases when the amplitude of the free surface oscillations is less than in a classical medium, that is strictly proved. Generally, this property is valid for an almost arbitrary set of physical parameters, and only for exclusive combinations of parameters can one expect inverse wave nature.

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